Generic occurrence of rings in rotating systems

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Abstract

In rotating scattering systems, the generic saddle-center scenario leads to stable islands in phase space. Non-interacting particles whose initial conditions are defined in such islands will be trapped and form rotating rings. This result is generic and also holds for systems quite different from planetary rings.

Key words: Rings; rotating scattering systems; stable orbits; saddle-center bifurcations.
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Light particles interacting with massive rotating systems are frequently encountered in diverse fields of physics. Electrons in rotating molecules and particular versions of the restricted three-body problem in celestial mechanics are the most well-known examples, but some nuclear models fall into the same category. Situations with large angular momenta will be of particular interest. Increasing amounts of information about narrow planetary rings suggest that such rings are often associated with the so-called shepherd satellites [1], and may exist due to mechanisms somewhat more complicated than the well known broad rings [2,3]. The implications of such mechanisms for the above mentioned systems could be far reaching.

The purpose of this paper is to show that there exists in rotating systems a generic mechanism to obtain narrow rings with structure, that does *not* depend on Kepler orbits. The genericity of the mechanism guarantees that the stable orbits supporting the ring structure are rather insensitive to small

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perturbations and thus may play a role in different situations of the type mentioned above.

The principal result of the paper is a very general argument to support the occurrence of such rings in terms of one of the two generic scenarios for the formation of bounded trajectories in a scattering problem as a function of some external parameter. We exemplify this argument using the simple model of discs rotating around a center outside of these discs, which we proposed earlier [4]. We shall find a large number of sometimes complicated rings. The introduction of a second disc on a smaller orbit may limit these to a small number of narrow rings. Our line of argumentation does not require that both discs move at the same angular velocity to maintain a ring structure. This implies that we leave the framework of a rotating system and with it the conservation of the Jacobi integral; the generic properties play an important role in the transference of the results obtained to situations for which the Jacobi integral is not conserved.

If we transform free motion to a synodic frame (rotating frame) the Jacobi integral becomes the Hamiltonian and the motion becomes a two-armed spiral. One arm for the incoming motion up to the point of closest approach to the center of rotation, and another for the outgoing motion after this point. We next consider a convex repulsive potential rotating at some distance from the center. It is clear that this potential can throw a particle from the outgoing arm of one spiral to the incoming arm of another. If the radial motion of the particle is sufficiently slow that the particle can hit the same repulsive potential again on its new way out, the trajectory may be confined. This will occur if the absolute value of the Jacobi integral is sufficiently small to avoid that the particle will always escape after at most one collision. Thus by gradually reducing (or increasing for negative values) the Jacobi integral we will find bounded orbits at some value. There are two generic ways for this to occur: First, a saddle-center bifurcation which will always produce a stable and an unstable periodic orbit (elliptic and hyperbolic fixed points) [5]. Second, the scenario where abruptly a fully hyperbolic structure appears in phase space [6]; the latter are typically associated with maxima in the potential and what is known as orbiting scattering.

We shall concentrate on the first scenario, for which the occurrence of stable periodic orbits is generic. In the synodic frame non-interacting particles dispersed along the corresponding stable island will usually form an eccentric ring obtained from the spiral orbit deformed by the potential such as to form a closed path. In the sidereal (space-fixed) frame this ring will undergo a precession corresponding to the frequency of the rotation. As we shall see below, the ring obtained in the sidereal frame may be quite different from the actual trajectories of the individual ring particles.
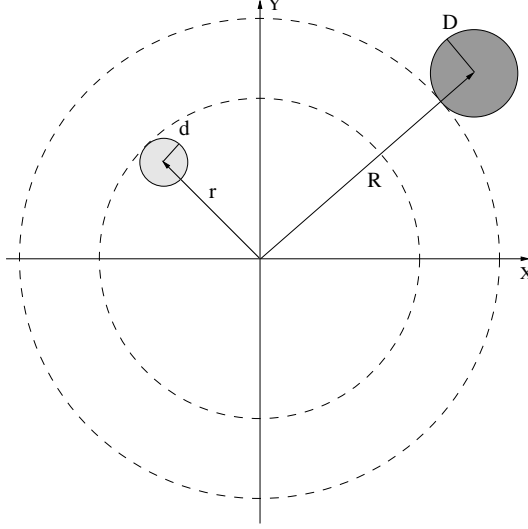
Fig. 1. Geometry of the two rotating discs scattering billiard: R and r define the radial position of the discs; D and d their radii. Both discs rotate about the origin with frequencies Ω and ω.

To illustrate the general argument we shall start from a toy-model, which has been studied before [4]. The repulsive potentials in this case corresponds are two (rotating) hard-wall scatterers. This model has the additional advantage of excluding the second generic scenario mentioned above.

We consider first one disc rotating around a center which lies outside this disc (Fig. 1); the motion will be restricted to a plane. We shall denote by R the radial position of the center of the disc with respect to the center of rotation, its radius by D $(R > D)$ and the angular velocity by Ω. In the sidereal frame, particles move on straight lines with constant velocity until they collide with the disc; otherwise they escape. A collision with the disc will typically change the direction and the magnitude of the velocity of the particle. Only if the collision is radial, i.e. in one of the intersection points on the disc of the line that joins the center of rotation (origin) with the center of the disc, the magnitude of the velocity will be unchanged, and the incoming collision angle is equal to the outgoing one. For these orbits, we can choose the outgoing angle α and the velocity of the particle to obtain identical and consecutive radial collisions. In the synodic frame, these orbits are periodic and symmetric. Such orbits provide the backbone of the horseshoe construction [7], as was shown for this model in Ref. [4]. In terms of the Jacobi integral, these orbits are given by

$$J_n = 2\Omega^2(R - D)^2 \frac{\cos^2 \alpha - [(2n + 1)\frac{\pi}{2} - \alpha] \sin 2\alpha}{[(2n + 1)\pi - 2\alpha]^2}. \tag{1}$$

Here, α is measured with respect to the normal at the radial collision point, and n is the number of complete rotations the disc performs between consecutive
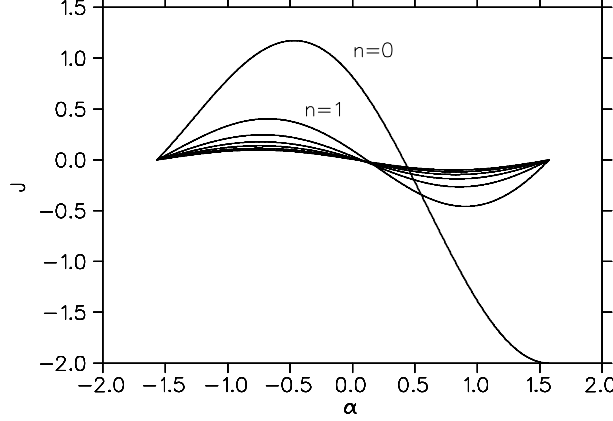
Fig. 2. Chart of symmetric periodic orbits of the rotating disc billiard for various values of n ($R = 3$, $D = 1$ and $\Omega = 1$).

collisions.

In Fig. 2 we show for small n the characteristic curves J_n for the symmetric periodic orbits. As mentioned above, there is a connected interval of the Jacobi integral where the action of the repulsive potential can build periodic orbits. In fact, by reducing the absolute value of the Jacobi integral, a saddle-center bifurcation creates a pair of periodic orbits every time a maximum or minimum is crossed. One of these is stable and the other unstable [4]. In the neighborhood of every stable periodic orbit there exist small regions of stability, which allow for the appearance of rings as described above.

As the stability of the symmetric periodic orbits is known [8], we can actually determine at what angle α stable orbits, periodic in the synodic frame, can occur. Indeed, for each n there is exactly one prograde ($\alpha > 0$) and one retrograde orbit ($\alpha < 0$), that will be stable over some interval of decreasing $|J|$ and then undergo a period doubling bifurcation sequence. The only exception is the prograde $n = 0$ solution which is marginally stable for a single J at $\alpha = \pi/2$. The angles $\alpha_n^\pm$ for which each saddle-center bifurcation occurs are given by

$$\tan \alpha_n^\pm = \frac{2}{(2n+1)\pi - 2\alpha_n^\pm} \pm 1. \tag{2}$$

Here, the upper index identifies the sign of the solution of Eq. (2), that is whether the solution corresponds to a prograde or retrograde trajectory. The absolute value of the solutions of this equation are shown in Fig. 3. The stable periodic orbits are found at absolute values of angles slightly larger than these [8].

Ring structures are obtained if we distribute initial conditions randomly in the interaction region and observe those that remain after a long time (Figs. 4).
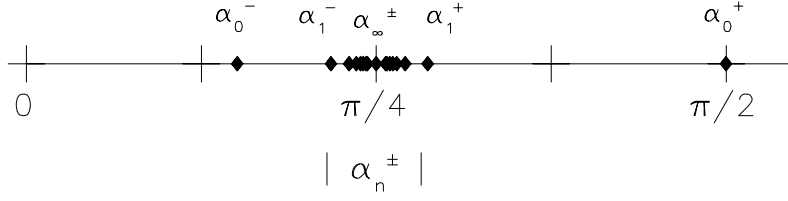
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Fig. 3. Absolute value of the solutions of Eq. (2), giving the angles $\alpha_n^{\pm}$ where new periodic orbits are created by saddle-center bifurcations. The stable periodic orbits will appear for values slightly to the right of the points shown.

Here, we have assumed that ring particles do not interact among themselves. If we consider all these stability regions together, we find a rather large area of rings that can coexist if there is no restriction on the initial conditions. This would lead to a wide ring with very complicated structure. Each stable region contributes a narrow ring that may have several loops, giving rise to different strands.

One way to obtain well defined narrow rings with only their intrinsic structure would be to limit initial conditions to the surroundings of one of the stable islands (as we often do for numerical purposes in our Monte Carlo calculations). Yet we do not want to argue such a selection, as this would imply information about the formation of rings, which is by no means the subject of this Letter. Based on the known presence of two shepherds near some narrow rings [1], we introduce a second disc moving on a circular trajectory with respect to the same center that lies inside the one we have considered above (see Fig. 1). We shall proceed to show that a second disc indeed provides such a selection mechanism.

The main effect of a second disc on a smaller orbit will be to sweep many of the possible stable orbits. The corresponding elliptic regions and therefore the associated rings will disappear. Clearly, new ones will be created inside the inner edge of the inner disc, but those are again of the type just discussed and we shall disregard them. Also, new periodic orbits involving collisions with both discs will show up, but these tend to be very unstable. We shall thus concentrate on the periodic orbits of the outer disc that are not affected by the inner one.

The simplest case is, in some sense, the one where the two discs move with incommensurable frequencies, although this implies that the Jacobi integral is no longer conserved. We proceed to evaluate which periodic orbits will not be affected by the inner disc, whose center is at distance r from the rotation point and whose radius is d. We assume that the orbits of the discs are non-overlapping, i.e., $r + d < R - D$. From the geometrical arrangement, it is clear that all the orbits that cross the outer edge of the inner disc will be affected by this disc. In terms of the angle α, we find that the orbits that will not be
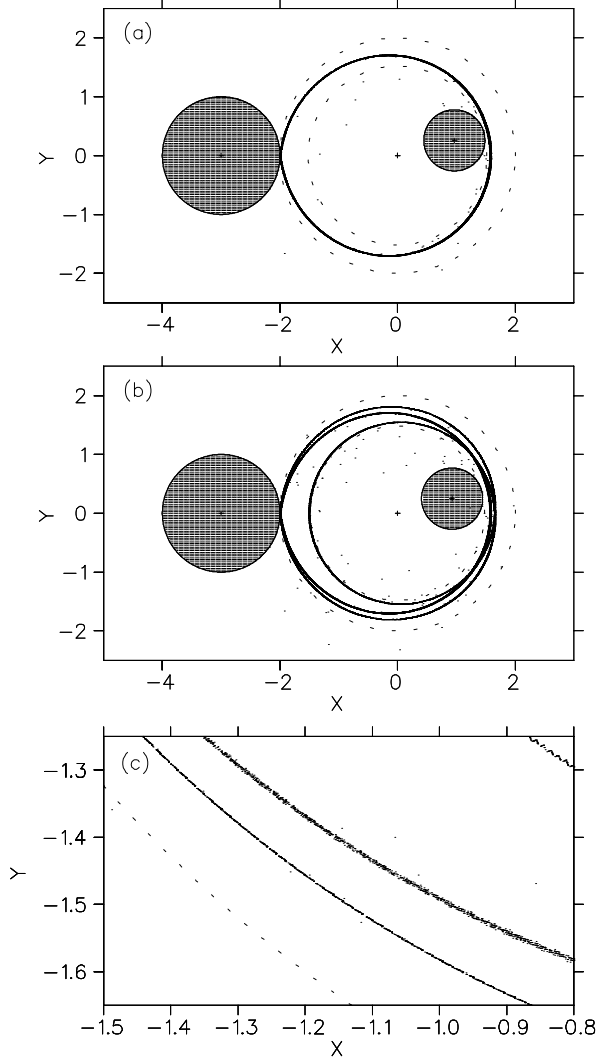
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Fig. 4. Rings of the two-disc billiard in the sidereal frame (R and D chosen as in Fig. 2). (a) Only the α_1^+ component survives ($r = 1$, $d = 0.52$); (b) the α_2^+ component is also present ($r = 0.96$, $d = 0.52$). (c) Detail of (b) showing the finite width of the ring.

affected satisfy

$$|\alpha| \geq \alpha_{\max} = \arcsin \frac{r + d}{R - D}. \qquad (3)$$

Note that for commensurable and in particular equal frequencies, resonant conditions may preserve some ring components for special configurations, while for other configurations these components may be wiped out. Yet, for the survival of rings fulfilling Eq. (3), no resonance condition is necessary.

Equation (3) permits to predict, which components will be unaffected by the inner disc. In this sense, Eq. (3) and the precise positions of $\alpha_n^\pm$ define selection
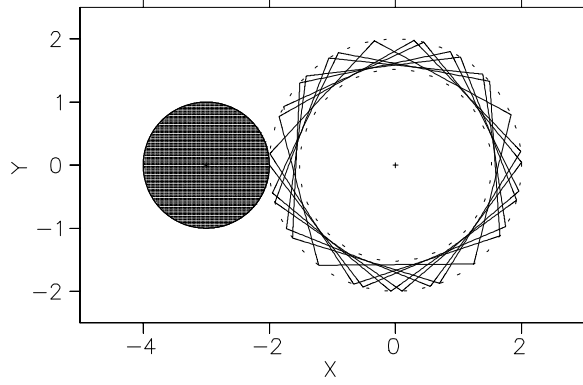
Fig. 5. Trajectory in the sidereal frame of a ring particle corresponding to α_1^+. The initial conditions were set in the position where the disc is shown. The dotted circles represent the paths drawn by the inner and outer edges of the outer and inner discs respectively.

rules. For instance, if the geometry is such that the condition $\alpha_2^+ < \alpha_{\max} < \alpha_1^+$ holds, the system will only display one ring corresponding to the α_1^+ stable region (see Fig. 3). For $\alpha_{\max}$ sufficiently larger than α_1^+, rings will not occur in the system, while for $\alpha_{\max} < |\alpha_0^-|$ all possible strands will show up. We conclude that ring components corresponding to prograde orbits are more likely to be found than those associated to retrograde orbits.

In Fig. 4a we present in the sidereal frame, examples of the ring structures found when only the α_1^+ component survives, and in Fig. 4b the case when the α_2^+ ring is also present. Figure 4c shows the finite width of the rings in an amplified region. As a function of time, these rings rotate with the frequency of the outer disc. While these figures show rings at fixed time in the sidereal frame, the individual particles follow polygonal orbits corresponding to reflection angles near a_n^+, as shown in Fig. 5. These polygons will typically not close. Note that the corresponding periodic orbits in the synodic frame will have a shape similar to the rings in the sidereal frame.

If we would use a mountain-like potential rather than a hard disc, associated with the hilltop the second type of scenario, an abrupt bifurcation, may occur [6]. This scenario implies the sudden appearance of a hyperbolic structure which is structurally stable. But in these case pruning would typically set in after a finite change of the Jacobi integral and we revert to the other scenario if the mountain has a steep slope.

For attractive potentials we can have other periodic orbits, but typically we would expect that the ones we consider still exist, and follow a similar scenario. The case of attractive gravitational potentials is of particular interest: the fact that the central potential will produce elliptic or hyperbolic trajectories in the sidereal system can easily be included in the argument. On the other hand, the existence of the weaker $1/r$ potentials from the shepherd

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moons seems to fit only loosely in the picture discussed above. Yet, we may recall that Hénon [9] found that the periodic orbits of consecutive collision in the restricted three-body problem for zero mass parameter (the Kepler problem with a non-attractive rotating singularity) determine the structure of the periodic orbits for finite small masses. This obviously carries over to the four-body problem. Therefore, we can expect the generic hard-disc results to have some qualitative similarity to the shepherd situation, except that the roles of the inner and outer shepherds may be interchanged [10]. This argument is further supported by some numerical investigations in the restricted three-body problem for small mass ratio [11], as well as by the recent rigorous proof, for the same problem, of the existence of a chaotic subshift near collision orbits [12]. Further research in this direction is on the way.

Finally we would like briefly to touch upon the structure of the narrow rings we found. As mentioned above and illustrated in Figs. 4, stable periodic orbits belonging to different stability regions lead to independent ring components, each of which may display several loops ($n+1$ for retrograde and n for prograde orbits). Particles in different rings will clearly have different speeds. There is a second mechanism that does not imply such a difference of speeds. A single strand will generically become structured, when the elliptic fixed point undergoes the period doubling cascade. Just after the period doubling, the ring component will have first two, then four, etc., entangled strands associated with each region of stability. This mechanism can produce a narrow braided ring. In this case ring particles in the different strands move almost synchronously. Indeed, the relative motion of ring particles and system rotation is a result of the particular potential and may be near zero for the first stable prograde orbit.

In conclusion, we have established the existence of a generic scenario for the occurrence of stable rings of non-interacting particles in rotating systems.

The example we discussed above is particular in several senses: First, we exclude the abrupt bifurcation scenario. If instead of hard discs we consider smooth potential hills of a Gaussian or similar shape, this scenario will occur at the hill tops, and give rise to interesting phenomena. Yet in their steep flanks the usual saddle-center bifurcations will still take place. Second, we violate the invariance of the Jacobi Integral only in a subspace of phase space, that we are not interested in. If we think of semi-classical applications, the border of the invariant subspace will be smeared out and the symmetry breaking will become ubiquitous though it may be weak in some parts. It is here that the genericity of saddle-center bifurcations and their consequent insensitivity becomes very important, as it guarantees that the structures survive with minor changes. Third, the rings, as found here, display complicated structure, both because two or several rings with different particle speed may coexist and interfere in complicated ways in semi-classics, and because a single ring

may have strands of similar particle speed as the central orbit undergoes a period-doubling cascade.

In semi-classics we can proceed to perform the calculation in the rotating system, where we will use standard techniques both for the stable and unstable orbits; thereafter, we can transform the resulting wave function to space fixed system. The errors of the semi-classical approximation will be modified, but the result in the space fixed frame will be correct to the same order in $\hbar$. Thus we can expect to see such structures whenever the motion of the light particle is too slow for the Born-Oppenheimer approximation to be valid, but semi-classical reasoning is adequate. This will *e.g.* be the case for very large angular momenta both in molecules and Nuclei.

In this letter we emphasize the generic character of the result obtained. The possible applications have to be studied within each particular field. In most cases the fact, that we expect the same behavior for attractive forces is very important. In nuclei, for example, the rotating mean potential may well result in resonances of surface nucleons that spend most of the time outside this potential. In molecules high angular momenta for the core are still quite inaccessible to most experiments, but it will be interesting to see what happens at the verge of the formation of Rydberg states in such cases. Finally we should not discard the possibility that such effects are relevant to argue that narrow rings with shepherds may live longer than the broad rings due to the generic stability of such structures [10].

References

[1] P. Goldreich and S. Tremaine, Nature **277**, (1979) 97.

[2] R. Greenberg and A. Brahic (eds.), *Planetary rings.* The University of Arizona Press, Tucson (1984).

[3] B.A. Smith, and the Voyager imaging team, Science **212**, (1981) 163; **215**, (1982) 504.

[4] N. Meyer *et al.*, J. Phys A: Math. Gen. **28**, (1995) 2529.

[5] M. Ding *et al.*, Phys. Rev. A **42**, (1990) 7025; Phys Lett. A **153**, (1991) 21.

[6] T. Tél, C. Grebogi and E. Ott, Chaos **3**, (1993) 495.

[7] L. Benet *et al.*, Physica D **131**, (1999) 253.

[8] H.R. Dullin, Nonlinearity **11**, (1998) 151.

[9] M. Hénon, Bull. Astron. (serie 3) **3**, (1968) 377; *Generating Families in the Restricted Three-Body Problem*, Lecture Notes in Physics **m52**, Springer, New York (1997).

[10] L. Benet and T.H. Seligman, Prog. Theo. Phys. Supl. (2000), in press.

[11] L. Benet, T.H. Seligman and D. Trautmann, Celest. Mech. Dynam. Astron. **71**, (1999) 167.

[12] S.V. Bolotin and R.S. MacKay, "Periodic and chaotic trajectories of the second species for the n-centre problem", preprint.